Workshop of the German Actuarial Academy

Schloß Reisensburg
Günzburg, Germany

August 31 and September 1, 2004

Workshop on Asset Liability Management in Insurance

September 23–25, 2004, Vienna, Austria

Organized jointly by

- Vienna University of Technology
- Austrian Financial Market Authority
- University of Applied Sciences BFI in Vienna
- Actuarial Association of Austria
- Vienna University of Economics and Business Administration
- Scientific Association "Insurance, Financial, and Operational Risk Management"

http://alm.fam.tuwien.ac.at/

Introduction to Credit Risk Modelling

Workshop of the German Actuarial Academy
Schloß Reisensburg, Günzburg
10.00–11.30, August 31, 2004

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Components of Credit Risk

- Arrival risk: Uncertainty whether a default will occur or not. Measured by the probability of default, within a given time horizon, usually one year.
- Timing risk: Uncertainty about the time of default.
- Exposure risk: Relatively clear for loans or bonds (face value, market value), greater uncertainty in the credit reinsurance business as primary insurers might have successfully decreased credit lines in advance.
- Recovery risk: Uncertainty about the size of the loss w.r.t. the exposure. Historical data show a large variability of recovery rate, depending on collateral, seniority of the bond, etc. Specified by conditional distribution of recovery rate given default occurred.

Components of Credit Risk (Cont.)

- Rating transition risk: Risk of changing market price of a defaultable security due to a changed perception of the market towards the timing or recovery risk (without an actual default already happening). It often happens together with an up- or down-rating of the creditworthiness by a rating agency.
- Default correlation risk: Risk of several obligors defaulting together; leads to substantial losses even in well diversified portfolios. Defaults in investment-grade rating classes are rare, hence it is hard to collect data to estimate the dependence of defaults.

Time Series versus Default Modelling

Time series modelling (e.g. exchange rates)

- Collect data for a long time (e.g. CHF/US-$).
- Assume stationarity of stochastic behaviour, fit a suitable model (random walk, GARCH, etc.) and make predictions about the future.

Default modelling

- Observing a firm until today doesn't give a default observation (→ observation bias).
- Solution: Observe a group of firms, draw conclusions for a specific firm.
- Problems: Relevance of data for the specific firm? When are firms similar w.r.t. creditworthiness?

Credit Ratings for Bonds

A credit rating is a current opinion of an obligor's overall financial capacity (its creditworthiness) to pay its financial obligations.

Standard & Poor's Investor Services

Investment grade: AAA, AA, A, BBB
Speculative: BB, B, CCC, CC, C (D = Default)
AA–B: + = above, − = below average in rating class

Moody's Investor Services

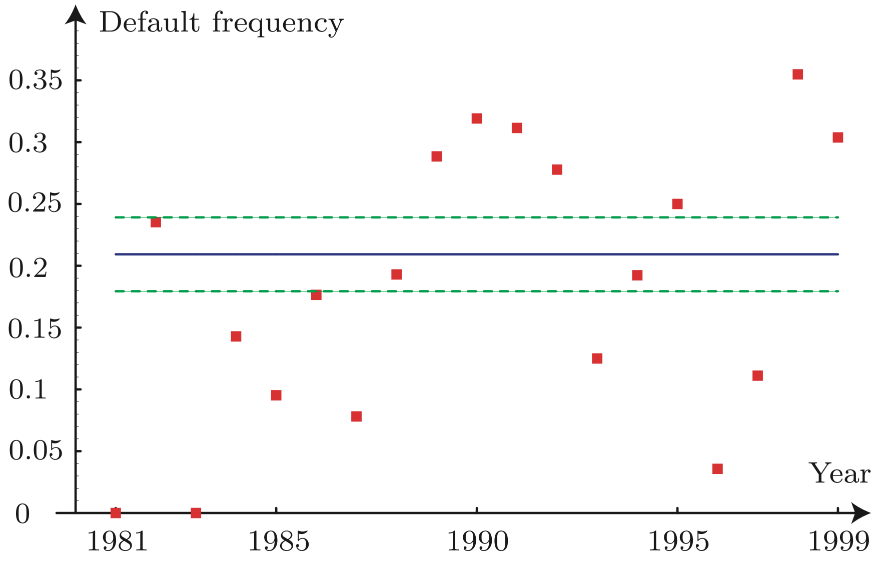
Investment grade: Aaa, Aa, A
Speculative: Baa, Ba, B, Caa, Ca, C
Aa–B: 1 = above, 2 = at, 3 = below average in rating class

Some Offered Bonds (in CHF)

Company	Coupon	Maturity	Price	Rating		Date
				S & P	Moody's	
WestLB[*]	float	28.7.06	100.060	AA	Aa2	22.7.
GECC[†]	float	23.7.07	100.070	AAA	Aaa	20.7.
Commonwealth Bank of Australia	1.750	4.9.07	100.085	AA–	Aa3	30.7.
Principal Financial Global Funding	2.750	12.7.10	100.190	AA	Aa3	8.7.
Sigma Finance	2.375	29.7.11	100.225	AAA	Aaa	21.7.
Hypo Tirol Bank	3.000	20.11.12	101.600	AAA		5.8.

[*] 3-month Libor flat
[†] 3-month Libor + 2 bp

Neue Zürcher Zeitung (NZZ, Vol. 225), July 7, 2004

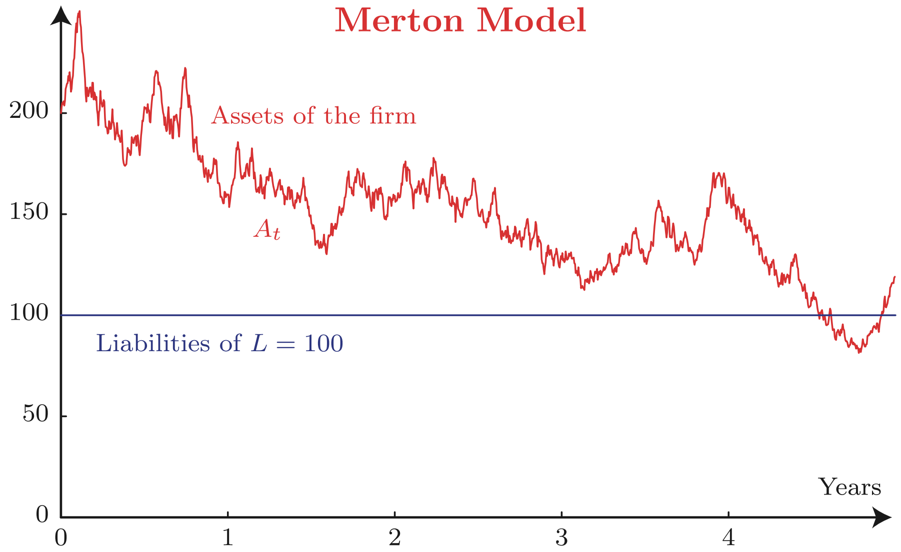


Empirical default frequencies l_i/n_i of the CCC-rated firms with ML-estimate $\hat{p}$ and asymptotic 95%-confidence interval.

Classification of Credit Risk Models

- Firm-value (or structural) models
  Pioneered by Black & Scholes (1973), Merton (1974)
  Industry models: Portfolio Manager (by KMV),
  CreditMetrics (RiskMetrics Group)
- Intensity-based (or reduced-form) models
  Jarrow & Turnbull (1995),
  Jarrow, Lando & Turnbull (1997),
  Lando (1996, 1998), Duffie & Singleton (1999)
- Actuarial models
  Mixture models, CreditRisk$^+$ (CS Financial Products)
- Macroeconomic models
  Industry: CreditPortfolioView (McKinsey & Company)

Merton Model



Exponential Brownian motion $A_t = 200\exp(0.3W_t)$.
Default occures when $A_t < L$.

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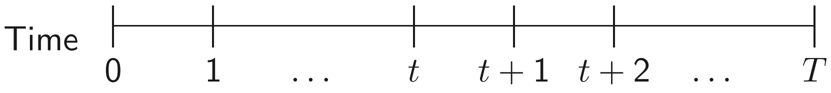
Discrete-Time Motivation for Intensity Models



Notation

X	promised (but defaultable) payout at time T
h_t	conditional probability at time t for default during the period $[t,t+1]$
r_t	continuously compounded, default-free interest rate for the period $[t,t+1]$
φ_{t+1}	random recovery at $t+1$ in case of default during $[t,t+1]$
$\mathbb{E}_{\mathbb{Q}}[\,\cdot\,\vert\mathcal{F}_t]$	conditional expectation under $\mathbb{Q}$ given all the information $\mathcal{F}_t$ up to time t

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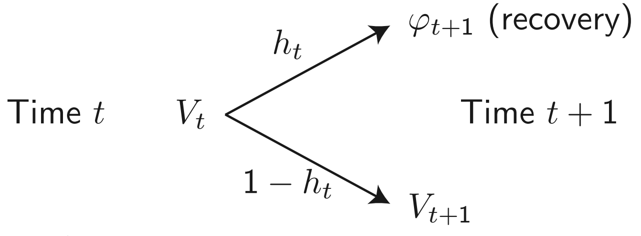
Evolution of Market Value V_t



Recursion formula

$$V_t = h_t\, e^{-r_t}\, \mathbb{E}_{\mathbb{Q}}[\varphi_{t+1}|\mathcal{F}_t] + (1-h_t)\, e^{-r_t}\, \mathbb{E}_{\mathbb{Q}}[V_{t+1}|\mathcal{F}_t]$$

with terminal value $V_T = X$.
An explicit formula for V_0 by backward induction is available but complicated to evaluate.

Different Assumptions for the Recovery

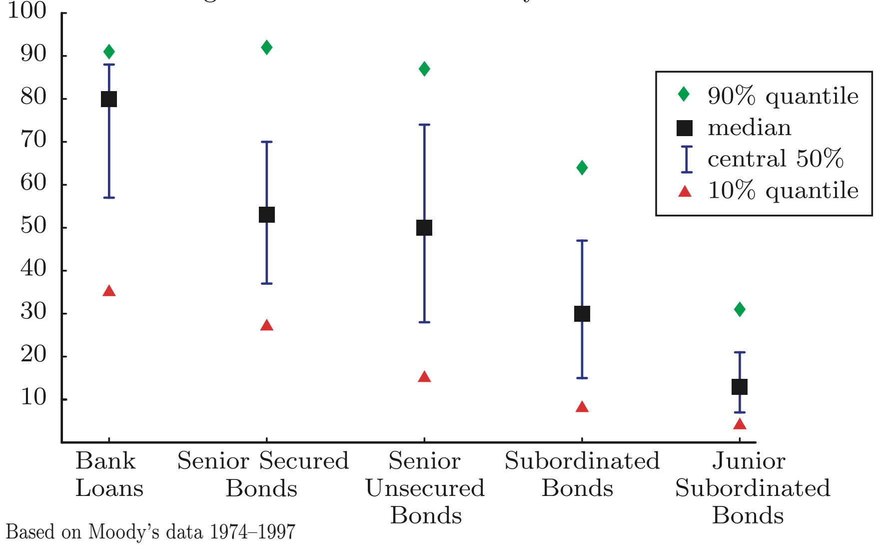
Recovery of face value: $\varphi_t = 1 - L_t$
The creditor receives a (possible random) fraction of the face value 1 immediately upon default.

Recovery of treasury: $\varphi_t = (1-L_t)P(t,T)$
The creditor receives a (possible random) fraction of a corresponding default-free government bond.

Recovery of market value (RMV):
$\mathbb{E}_{\mathbb{Q}}[\varphi_{t+1}|\mathcal{F}_t] = (1-L_t)\,\mathbb{E}_{\mathbb{Q}}[V_{t+1}|\mathcal{F}_t]$
The expected recovery is a (random) fraction of the expected market value in case of no default.

Distribution of Recovery by Seniority

Percentage of face value in secondary market after default



Based on Moody's data 1974–1997

Transition to Hazard and Loss Rates

With recovery of market value

$$V_t = \underbrace{\{(1-h_t)\,e^{-r_t} + h_t\,e^{-r_t}(1-L_t)\}}_{=:\,e^{-R_t}}\,\mathbb{E}_{\mathbb{Q}}[V_{t+1}|\mathcal{F}_t]$$

with $e^{-R_t} = (1-h_tL_t)e^{-r_t} \approx e^{-(r_t+h_tL_t)}$, because

$$\Bigl(1-\frac{h_t}{n}L_t\Bigr)^n \to e^{-h_tL_t} \qquad n\to\infty.$$

n is the number of subdivisions per period. Hence

$$V_0 = \mathbb{E}_{\mathbb{Q}}\bigl[e^{-(R_0+\cdots+R_{T-1})}X\bigr].$$

RiskLab Project: Intensity-Based Non-Parametric Default Model for Residential Mortgage Portfolios

Swiss banks hold over 500 billion CHF in mortgages.
Data for 73 683 obligors of Credit Suisse used.
Default intensity tested for dependence on
– Regional unemployment rates,
– Fixed- or variable-rate mortgage product,
– Interest-rate changes,
– Divorce rates,
– Regional real estate price indices,
– Time-lags until default.
Reference: Paper (38 pages) by Enrico De Giorgi
http://www.risklab.ch/Papers.html#RMSRMMLP

Modelling Dependence of Defaults

- Firm-value (or structural) models (asset correlations, macroeconomic factors)
- Intensity-based (or reduced-form) models (macroeconomic factors, business cycle)
- Actuarial models, mixture models (random intensities, random default probabilities)
- Infectious defaults (M. Davis & V. Lo, 1999)
- Looping defaults, primary/secondary firms (R. Jarrow & Fan Yu, 2001)
- Hindering defaults

⟶ Subject of current research

Introduction to CreditRisk$^+$, Features

- Developed by Credit Suisse Financial Products.
- Actuarial model for the aggregation of credit risks.
- Based on the Poisson approximation of individual defaults and the divisibility of the Poisson distribution.
- Takes random exposures/recovery rates into account.
- Probability generating function ϕ_X of the credit portfolio loss X is available in closed form.
- Distribution of X can be calculated from ϕ_X with a numerically stable algorithm.

Project CreditRisk$^+$

- Background: 2. Basel Capital Accord of the Basel Committee on Banking Supervision ("Basel II")
- Research and development cooperation of
  - Research Group Financial and Actuarial Mathematics
  - Austrian Central Bank (OeNB)
  - Austrian Financial Market Authority (FMA)
- Aim: Supervision of credit risk in the portfolio of all ($\geq$ 900) Austrian banks
- Large single credit risks are reported individually
- Efficient method and numerically stable algorithm to calculate risk of credit portfolio
- Java implementation (Mag. Severin Resch)

Motivation: Bernoulli Model for Defaults

- Independent Bernoulli loss indicators

$$L_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults (within one year),} \\ 0 & \text{otherwise.} \end{cases}$$

- Default probability $p_i = \mathbb{P}(L_i = 1)$ for $i = 1, \ldots, m$.
- Random number of defaults $N = L_1 + \cdots + L_m$.
- Probability distribution for $n \in \{0, \ldots, m\}$

$$\mathbb{P}(N = n) = \sum_{\substack{I \subset \{1,\ldots,m\} \\ |I| = n}} \underbrace{\mathbb{P}\big(L_i = 1_I(i) \text{ for } i = 1, \ldots, m\big)}_{= (\prod_{i \in I} p_i) \prod_{i \in \{1,\ldots,m\} \setminus I} (1 - p_i)}$$

$$m = 1000,\ n = 100 \implies \binom{1000}{100} \approx 6.4 \times 10^{139} \text{ terms}$$

Motivation: General Bernoulli Mixture Model

- $(P_1, \ldots, P_m) \sim F$ random default probabilities
- Assume $\mathbb{P}(L_i = 1 \,|\, P_1, \ldots, P_m) \overset{\text{a.s.}}{=} \mathbb{P}(L_i = 1 \,|\, P_i) \overset{\text{a.s.}}{=} P_i$
- Assume independence of $L_1, \ldots, L_m$ given $P_1, \ldots, P_m$.
- Then for all $l_1, \ldots, l_m \in \{0, 1\}$

$$\begin{aligned} &\mathbb{P}(L_1 = l_1, \ldots, L_m = l_m) \\ &\quad = \mathbb{E}\big[\mathbb{P}(L_1 = l_1, \ldots, L_m = l_m \,|\, P_1, \ldots, P_m)\big] \\ &\quad = \mathbb{E}\Big[\prod_{i=1}^{m} P_i^{l_i} (1 - P_i)^{1 - l_i}\Big] \\ &\quad = \int_{[0,1]^m} \prod_{i=1}^{m} p_i^{l_i} (1 - p_i)^{1 - l_i} \, F(dp_1, \ldots, dp_m) \end{aligned}$$

Observations . . .

- Already the Bernoulli model with independent loss indicators has far too many terms for the calculation of the portfolio loss distribution in the general case.
- In the general Bernoulli mixture model, individual terms are too complicated to compute numerically.
- Different exposures and recovery rates are not even considered.

. . . and Conclusions

- Simplifying assumptions are necessary.
- Approximations need to be considered.

Divisibility/Additivity of the Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ be independent. Then $X+Y \sim \text{Poisson}(\lambda+\mu)$ because for every $n \in \mathbb{N}_0$

$$\mathbb{P}(X+Y=n) = \sum_{k=0}^{n} \underbrace{\mathbb{P}(X+Y=n\,|\,Y=k)}_{=\mathbb{P}(X=n-k|Y=k)=\mathbb{P}(X=n-k)} \mathbb{P}(Y=k)$$

$$= \sum_{k=0}^{n} e^{-\lambda} \frac{\lambda^{n-k}}{(n-k)!} e^{-\mu} \frac{\mu^k}{k!} = \frac{e^{-(\lambda+\mu)}}{n!} \underbrace{\sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} \mu^k}_{=(\lambda+\mu)^n}.$$

More on divisibility: F.R. Steutel, K.V. Harn: *Infinite Divisibility of Probability Distributions on the Real Line.* Pure and Applied Mathematics, Vol. 259 (2004), Dekker, ISBN 0-8247-0724-9.

Simple Poisson Model for Defaults

- Number L_i of defaults of obligor $i \in \{1,\dots,m\}$
- Assume $L_i \sim \text{Poisson}(\lambda_i)$ for all $i \in \{1,\dots,m\}$ (several defaults of an obligor possible).
- Assume independence of $L_1,\dots,L_m$.
- Random number of defaults $N = L_1 + \cdots + L_m$.
- $N \sim \text{Poisson}(\lambda)$ with $\lambda = \lambda_1 + \cdots + \lambda_m$, i.e.,

$$\mathbb{P}(N=n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{for all } n \in \mathbb{N}_0.$$

- $m = 20$, $\lambda_i = 0.2 \implies \mathbb{P}(N > 20) \le 2 \times 10^{-9}$.

Poisson Approximation

- $X_1,\dots,X_m$ independent default 0-1-indicators
- Intensity $\lambda = \sum_{i=1}^m p_i$ with $p_i = \mathbb{P}(X_i = 1)$
- Number of default events $W = \sum_{i=1}^m X_i$
- Total variation distance

$$d_{\text{TV}}(\mu,\nu) = \sup_{A \subset \mathbb{N}_0} |\mu(A) - \nu(A)|$$

Quality of Poisson approximation (Barbour/Hall, 1984):

$$d_{\text{TV}}\big(\mathcal{L}(W), \text{Poisson}(\lambda)\big) \le \frac{1-e^{-\lambda}}{\lambda} \sum_{i=1}^m p_i^2$$

For full proof with Stein–Chen method, see e.g. Barbour, Holst and Janson: *Poisson Approximation,* Clarendon Press (1992).

Poisson Approximation using Stein–Chen Method

For $W = \sum_{i=1}^m X_i$ and $N \sim \text{Poisson}(\lambda)$ with $\lambda = \mathbb{E}[W]$ we claim (Barbour/Hall, 1984)

$$\sup_{A \subset \mathbb{N}_0} \big|\mathbb{P}(W \in A) - \mathbb{P}(N \in A)\big| \le \frac{1-e^{-\lambda}}{\lambda} \sum_{i=1}^m p_i^2 .$$

For $A \subset \mathbb{N}_0$ and $l \in \mathbb{N}_0$ define

$$f_A(l) = 1_A(l) - \mathbb{P}(N \in A).$$

Then $f_{\mathbb{N}_0} = 0$, $\mathbb{E}[f_A(N)] = 0$ and

$$\mathbb{E}[f_A(W)] = \mathbb{P}(W \in A) - \mathbb{P}(N \in A).$$

Stein Equation for f_A and Its Solution

For $A \subset \mathbb{N}_0$ and $l \in \mathbb{N}_0$ define $g_A(0) = 0$ and

$$g_A(l+1) = \frac{\mathbb{P}(N \le l,\, N \in A) - \mathbb{P}(N \le l)\,\mathbb{P}(N \in A)}{\lambda\,\mathbb{P}(N = l)}.$$

Since $\lambda\,\mathbb{P}(N = l-1) = e^{-\lambda}\lambda^l/(l-1)! = l\,\mathbb{P}(N = l)$, g_A solves the Stein equation

$$\begin{aligned}&\lambda g_A(l+1) - l g_A(l)\\ &\quad = \frac{\mathbb{P}(N = l,\, N \in A) - \mathbb{P}(N = l)\,\mathbb{P}(N \in A)}{\mathbb{P}(N = l)} = f_A(l).\end{aligned}$$

Exercise: g_A is bounded and unique.

Key Estimate Using the Stein Equation

With $W = X_1 + \cdots + X_m$ and $W_i = W - X_i$ we get

$$\begin{aligned}\mathbb{E}[f_A(W)] &= \mathbb{E}[\lambda g_A(W+1) - W g_A(W)]\\ &= \underbrace{\lambda}_{=\sum_{i=1}^m p_i} \mathbb{E}[g_A(W+1)] - \sum_{i=1}^m \underbrace{\mathbb{E}[X_i\, g_A(W_i + X_i)\,|\,X_i = 1]}_{=\mathbb{E}[g_A(W_i+1)] \text{ by indep. of } W_i,\ X_i}\, p_i\\ &= \sum_{i=1}^m p_i\, \mathbb{E}[\underbrace{g_A(W+1)}_{=g_A(W_i+1) \text{ if } X_i=0, \text{ and } g_A(W_i+2) \text{ otherwise}} - g_A(W_i+1)\,|\,X_i = 1]\, p_i.\end{aligned}$$

Hence

$$\bigl|\mathbb{E}[f_A(W)]\bigr| \le \sup_{l\in\mathbb{N}} \bigl|\underbrace{g_A(l+1) - g_A(l)}_{=:\,\Delta g_A(l)}\bigr| \sum_{i=1}^m p_i^2.$$

Estimate of Increments Δg_A of the Stein Solution

Set $g_k = g_{\{k\}}$. Since $\Delta g_k(l) \le 0$ for $k \ne l$ (exercise), we have $\Delta g_A(l) = \sum_{k\in A} \Delta g_k(l) \le \Delta g_l(l)$. Since $f_A + f_{A^c} = f_{\mathbb{N}_0} = 0$, we have $g_A = -g_{A^c}$, hence $-\Delta g_A(l) = \Delta g_{A^c}(l) \le \Delta g_l(l)$. Finally,

$$\begin{aligned}\Delta g_l(l) &= g_l(l+1) - g_l(l)\\ &= \frac{\overbrace{1 - \mathbb{P}(N \le l)}^{=\mathbb{P}(N>l)}}{\lambda} + \frac{\overbrace{\mathbb{P}(N \le l-1)\,\mathbb{P}(N = l)}^{\le l\,\mathbb{P}(1\le N\le l)/\lambda}}{\underbrace{\lambda\,\mathbb{P}(N = l-1)}_{=l\,\mathbb{P}(N=l)}}\\ &= \frac{\mathbb{P}(N \ge 1)}{\lambda} = \frac{1 - e^{-\lambda}}{\lambda}.\end{aligned}$$

General Poisson Mixture Model

- $\Lambda_1, \ldots, \Lambda_m$ random default intensities
- Assume for every $i \in \{1, \ldots, m\}$ and $l_i \in \mathbb{N}_0$

$$\mathbb{P}(L_i = l_i\,|\,\Lambda_1, \ldots, \Lambda_m) \overset{\text{a.s.}}{=} \mathbb{P}(L_i = l_i\,|\,\Lambda_i) \overset{\text{a.s.}}{=} \Lambda_i^{l_i} e^{-\Lambda_i}/l_i!$$

- Assume independence of $L_1, \ldots, L_m$ given $\Lambda_1, \ldots, \Lambda_m$.
- Then for the number $N = L_1 + \cdots + L_m$ of defaults

$$\mathcal{L}(N\,|\,\Lambda_1, \ldots, \Lambda_m) = \text{Poisson}(\Lambda_1 + \cdots + \Lambda_m),$$

i.e., for all $n \in \mathbb{N}_0$,

$$\mathbb{P}(N = n) = \mathbb{E}\Bigl[\frac{(\Lambda_1 + \cdots + \Lambda_m)^n}{n!} e^{-(\Lambda_1+\cdots+\Lambda_m)}\Bigr].$$

Gamma-Poisson Mixture Model

- Assume $\Lambda_i = \lambda_i \Lambda$ for all obligors $i \in \{1, \ldots, m\}$, where Λ is gamma distributed with $\alpha, \beta > 0$.
  Density
  $$f_\Lambda(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0.$$
- Define $\lambda = \lambda_1 + \cdots + \Lambda_m$ and $p = \beta/(\beta + \lambda)$.
  Then for every $n \in \mathbb{N}_0$

$$\mathbb{P}(N = n) = \mathbb{E}\Big[\frac{(\lambda\Lambda)^n}{n!} e^{-\lambda\Lambda}\Big] = \binom{n+\alpha-1}{n} p^\alpha (1-p)^n.$$

$\Longrightarrow$ No. of defaults has negative binomial distribution!

Input Parameters of CreditRisk$^+$

- Number of obligors $m \in \mathbb{N}$.
- Basic loss unit $E > 0$.
- Exposure $\nu_i E$ with $\nu_i \in \mathbb{N}_0$ of obligor $i \in \{1, \ldots, m\}$.
- Number $K \in \mathbb{N}_0$ of "sectors"
  or non-idiosyncratic, independent default causes.
- One-year default probability $p_i \in [0, 1)$ of obligor i.
  Set default intensity $\lambda_i = p_i$ or $\lambda_i = -\log(1 - p_i)$.
- Susceptibility $w_{i,0} \geq 0$ of obligor i to idiosyncratic risk.
- Susceptibility $w_{i,k} \geq 0$ of obligor i to sector risk k.
  Condition: $w_{i,0} + \cdots + w_{i,K} = 1$ for $i \in \{1, \ldots, m\}$
- Relative default variance $\sigma_k^2 > 0$ of sector risk k.

Notation for Default Events and Sector Risk

- Number L_i of defaults of obligor $i \in \{1, \ldots, m\}$
- Defaults $L_{i,0}$ due to idiosyncratic risk
- Defaults $L_{i,k}$ due to sector default risk $k \in \{1, \ldots, K\}$
  $$L_i = L_{i,0} + \cdots + L_{i,K}$$
- Sector risks $\Lambda_1, \ldots, \Lambda_K$ given by independent, gamma distributed random variables with $\mathbb{E}[\Lambda_k] = 1$ and $\mathrm{Var}(\Lambda_k) = \sigma_k^2$.

Total credit portfolio loss X in units of E

$$X = \sum_{i=1}^m \nu_i L_i = \sum_{i=1}^m \sum_{k=0}^K \nu_i L_{i,k}$$

Distributional Assumptions for CreditRisk$^+$

- $L_{i,0} \sim \mathrm{Poisson}(\lambda_i w_{i,0})$ for all obligors $i \in \{1, \ldots, m\}$.
- $L_{1,0}, \ldots, L_{m,0}$ are independent and independent from all other random variables.
- For all obligors i and sector risks $k \in \{1, \ldots, K\}$
  $$\mathcal{L}(L_{i,k} | \Lambda_1, \ldots, \Lambda_K) = \mathcal{L}(L_{i,k} | \Lambda_k) = \mathrm{Poisson}(\lambda_i w_{i,k} \Lambda_k).$$
- Conditional on $\Lambda_1, \ldots, \Lambda_K$, the family
  $$\big\{ L_{i,k} \mid i \in \{1, \ldots, m\}, k \in \{1, \ldots, K\} \big\}$$
  is independent.

Probability Generating Function for CreditRisk$^+$

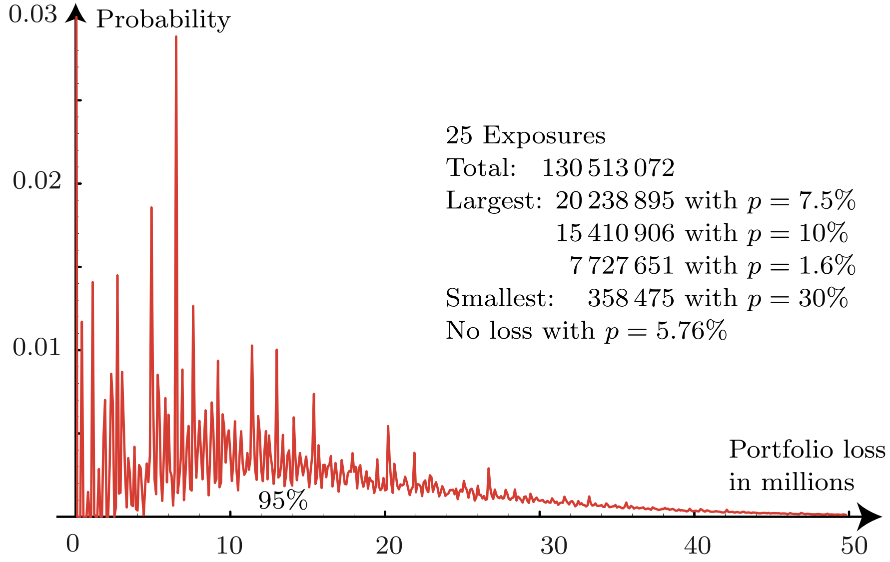
$$\phi_X(s) = \sum_{n\in\mathbb{N}_0} s^n\,\mathbb{P}(X=n) = \mathbb{E}\big[s^X\big]$$

$$= \prod_{i=1}^{m} \mathbb{E}\big[s^{\nu_i L_{i,0}}\big] \prod_{k=1}^{K} \mathbb{E}\big[s^{\sum_{i=1}^{m}\nu_i L_{i,k}}\big]$$

$$= \prod_{i=1}^{m} \exp\big(\lambda_i w_{i,0}(s^{\nu_i}-1)\big)$$

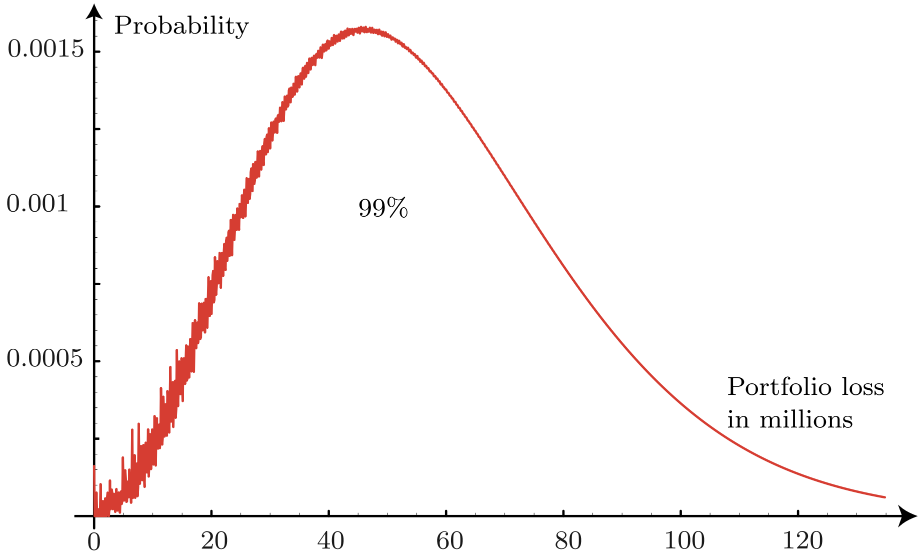
$$\times \prod_{k=1}^{K} \Big(1-\sigma_k^2 \sum_{i=1}^{m} \lambda_i w_{i,k}(s^{\nu_i}-1)\Big)^{-1/\sigma_k^2}$$

Inversion: H. Haaf, O. Reiß, J. Schoenmakers, *Numerically Stable Computation of CreditRisk*$^+$, Preprint, Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin (2003).

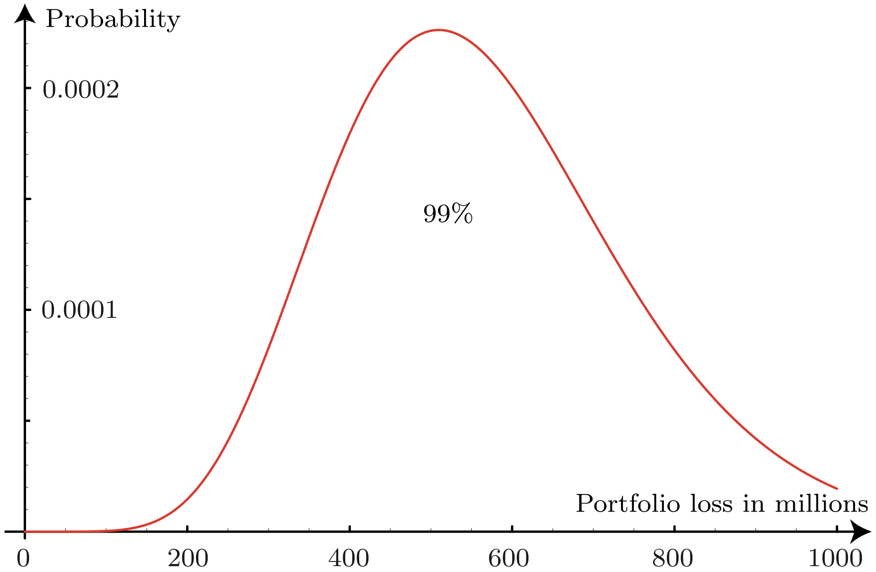


25 Exposures
Total: 130 513 072
Largest: 20 238 895 with $p = 7.5\%$
15 410 906 with $p = 10\%$
7 727 651 with $p = 1.6\%$
Smallest: 358 475 with $p = 30\%$
No loss with $p = 5.76\%$

Loss distribution in a credit portfolio of 25 exposures calculated with CreditRisk$^+$ and basic loss unit 100 000.



Loss distribution in a credit portfolio of 100 exposures calculated with CreditRisk$^+$ and basic loss unit 100 000.



Loss distribution in a credit portfolio of 1000 exposures calculated with CreditRisk$^+$ and basic loss unit 100 000.

Extensions of CreditRisk$^+$

- Random exposures/recoverables:
  Let $\{\nu_{i,j,k}\}_{j\in\mathbb{N}}$ be i.i.d. losses in units of E for defaults of obligor $i \in \{1,\dots,m\}$ due to sector risk $k \in \{0,\dots,K\}$. Total loss in credit portfolio

$$X = \sum_{i=1}^{m}\sum_{k=0}^{K}\sum_{j=1}^{L_{i,k}} \nu_{i,j,k}\,.$$

- Dependent sectors:
  Multiply the shape parameter of the gamma distributed $\Lambda_1,\dots,\Lambda_K$ by an independent gamma distributed S with $\mathbb{E}[S]=1$ and $\mathrm{Var}(S)=\sigma^2$.
- Calculation of risk contributions

Literature on Credit Risk Modelling

- Tomasz R. Bielecki and Marek Rutkowski, *Credit Risk: Modeling, Valuation and Hedging,* Springer (2003).
- Ch. Bluhm, L. Overbeck and Ch. Wagner, *An Introduction to Credit Risk Modeling,* Chapman & Hall/CRC (2003).
- Georg Bol et al. (Eds.), *Credit Risk, Measurement, Evaluation and Management,* Physica-Verlag, (2003).
- D. Duffie and K.J. Singleton, *Credit Risk: Pricing, Measurement and Management,* Princeton Univ. Press (2003).
- Bernd Schmid, *Pricing Credit Linked Financial Instruments: Theory and Empirical Evidence,* Lecture Notes in Economics and Math. Systems, Vol. 516, Springer (2002).
- Philipp J. Schönbucher, *Credit Derivatives Pricing Models. Models, Pricing and Implementation,* Wiley (2003).
- M. Gundlach and F. Lehrbass (Eds.), *CreditRisk$^+$ in the Banking Industry,* Springer (2004).